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Numerical calculations of the airflow over a ridge for small amplitudes

by

J. S. Sawyer and M. K. Hinds

(Meteorological Office)

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CORRIGENDUM

Meteorological Research Paper 1041 - Numerical calculations of the airflow over a ridge for small amplitudes, by J. S. Sawyer and M. K. Hinds.

Later studies have revealed that the calculations in this paper contain large errors due to the finite difference method used (truncation errors). The results should not therefore be quoted.

H W L Ab...

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Numerical calculations of the airflow over a ridge for small amplitudes

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1. Introduction

The mathematical problem of determining the airflow over hilly country has been considered by several authors. In order to make the mathematics tractable, some simplifying assumptions and approximations must be made, or the mathematical treatment must be limited to special cases. The more important simplifications which have commonly been made in the treatment of this problem are:-

(a) steady, non-turbulent flow
(b) neglect of viscous forces
(c) adiabatic conditions (without condensation effects)
(d) two-dimensional motion (i.e., motion over a ridge)
(e) no separation of the flow from the ground
(f) small amplitude disturbances (permitting the neglect of non-linear terms in the governing equations)
(g) specially simple variation of the wind and stability with height in the basic air stream
(h) horizontal scale small enough to neglect the earth's rotation.

It is not known how far such restrictions imposed on the mathematical problem will preclude direct comparison between calculated airflow and that observed in the atmosphere; nor is it clear which is the most important restriction to lift if any more complicated mathematical formulation of the problem is to be attempted.

The earlier work of Lyra (1943) and Queney (1947) was restricted to currents in which the static stability and wind speed in the basic current were the same at all levels. However the treatment given by Scorer (1949) clearly shows that the vertical variation of both the stability and the wind speed can have very large effects on the pattern of air motion over a barrier. There have been a number of published discussions of the flow in air streams in which either wind, stability or both varies with height but, in order to keep the mathematical analysis within bounds, the authors have been restricted to simple and often highly artificial air currents. Probably the most realistic formulation for which a solution has been obtained is that used by Scorer (1954) and Wallington (1955) in which the atmosphere is divided into three layers each with a constant value of Scorer's parameter "l".

It seems unlikely that any much more complicated variation of wind speed or stability with height could be treated conveniently by analytical methods, and the present paper treats the problem by the methods of numerical analysis and finite differences. This has the advantage that more complex vertical variations of wind and stability can be used; it also permits a realistic representation of the ground profile without undue complication. The equations have been

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linearised so that the results still apply only to small amplitude disturbances, but it seems likely that finite difference methods will also be required if any attempt is made to calculate the form of finite amplitude disturbances, and the present calculations provide indications of some of the problems which will arise.

2. The differential equation governing the motion

The appropriate perturbation equation governing the two-dimensional motion of a stably-stratified air stream on a scale appropriate to the meteorological problem has been derived in various forms, differing slightly according to the assumptions and simplifications made.

Scorer (1955) gives (his equation 7.5)

$$\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + \left(\frac{g\beta}{U^2} - \frac{\partial^2 U / \partial z^2}{U} \right) w = 0 \qquad (1)$$

where x and z are the horizontal and vertical coordinates,
w is the vertical velocity,
U is the horizontal velocity in the undisturbed stream,
$\beta = \theta^{-1} \partial\theta/\partial z$ is a measure of stability (θ = potential temperature).

The basic equation used in the present work has essentially the same form. When numerical constants have been inserted to make the equation convenient for numerical manipulation, it is

$$\frac{\partial^2 \phi}{\partial p^2} + A \frac{\partial^2 \phi}{\partial x^2} + B\phi = 0 \qquad (2)$$

where $A = 2{\cdot}2811 \times 10^{-4} \theta^2 / p^{\frac{1}{4}}$

$$B = -\left\{ \frac{148{\cdot}16 \left[\partial\theta/\partial p \right]_0}{U^2 p^{\frac{1}{4}}} + \frac{\partial^2 U / \partial p^2}{U} \right\}$$

In this formulation of the problem, p (= pressure) is used as a vertical coordinate and the flow is defined by a stream function ψ such that $u = \partial\psi/\partial p$ and $\tilde{w} \equiv dp/dt = -\partial\psi/\partial x$. ϕ, which has to be determined from equation (2), is the departure of the stream function ψ from its value at the same pressure level in the undisturbed stream.

In deriving the numerical constants the unit of horizontal distance has been taken as 1 Km, and the unit of pressure 100 mb. U is expressed in m/s and θ in °A, the differentiation $\left[\partial\theta/\partial p \right]_0$ being carried out in the undisturbed stream. (Since $\partial\theta/\partial p < 0$, B is normally positive).

The derivation of equation (2) is given in Appendix I. The most important assumptions which have been made are:-

(a) steady, frictionless, adiabatic flow
(b) horizontal scale small enough to neglect the earth's rotation (i.e., less than 50 Km)
(c) accelerations small compared with gravity
(d) amplitude small enough to permit neglect of non-linear terms.

3. Boundary conditions

In order to carry out the solution of equation (2) it is necessary to specify adequate "boundary conditions" to render the solution unique. In the numerical solution of equations such as (2), it is usually possible to specify conditions based on the physical formulation of the problems which apply at its boundaries of the area of integration. This is not possible in the present problem because

no real physical boundaries exist except at the ground.

In order to obtain unique solutions of (2) the following conditions have been applied:-

(1) The disturbance of the stream function, $\phi \to 0$ as $x \to -\infty$. The flow is assumed to be in the positive direction of x and this condition implies that the disturbance dies away upstream from the ridge.
(2) The disturbance of the stream function should remain finite as $x \to +\infty$ (i.e., downstream from the ridge).
(3) The lowest stream line should follow the ground contour.
(4) The disturbance of the stream function vanishes (i.e., $\phi = 0$) at some specified level which is the upper boundary of the area of integration.

Requirements (1) and (2) are clearly plausible, although since the equations do not apply to disturbances of large amplitude a solution which appeared to grow indefinitely downstream might not in fact do so if the full non-linear equation could be solved.

Assumption (3) rules out any separation of the flow from the surface or any special consideration of a turbulent boundary layer. Nevertheless consideration of flow in which the flow follows the lower boundary without friction is a necessary preliminary to any more complex treatment of the lower boundary. As in the present report we are concerned only with small amplitude disturbances of the flow, at the lower boundary, p = 10 (1000 mb), we can write the boundary condition as

$$\phi = - g \rho U H(x)$$

where H(x) is the height of the lower boundary above its level at a distance up or downstream.

Assumption (4), which is equivalent to imposing a rigid upper lid on the atmosphere, is the most difficult to justify. Corby and Sawyer (1956) have shown that, if such a rigid lid is imposed and then withdrawn to infinity, the solution tends to a limit which is, presumably, the solution appropriate to the atmosphere. However, it appears that for a close approach to this limit, the boundary must be at a considerable height in the stratosphere - too great a height to provide a practical limit for numerical integration.

It is also apparent from Corby and Sawyer's treatment that the introduction of a "rigid lid" at a level in the upper troposphere may make possible lee-wave oscillations which would not be possible if stability conditions characteristic of the troposphere extended to an indefinite height. On the other hand, rather similar oscillations to those arising with a "rigid lid" are possible if the existence of a highly stable stratosphere is taken into account. In other words the tropopause acts in some degree like a rigid lid, so that the flow patterns calculated with boundary assumption (4) above may be reasonably indicative of the behaviour of the atmosphere, provided that the upper boundary is chosen to coincide with the tropopause. The significance of the oscillations arising from the upper boundary condition is further discussed in Appendix II.

4. Method of solution of the differential equation

In order to perform the numerical solution of equation (2) the problem is transformed to one of finite differences by replacing the differential coefficients by their finite-difference approximations, using a rectangular lattice with mesh length δx and δp for this purpose.

The finite difference equation to be satisfied at each point of the lattice is then

$$\phi_{r,s+1} + \phi_{r,s-1} + A_s \frac{(\delta p)^2}{(\delta x)^2} \left\{ \phi_{r+1,s} + \phi_{r-1,s} \right\} - \left\{ 2 + 2 A_s \frac{(\delta p)^2}{(\delta x)^2} - B_s (\delta p)^2 \right\} \phi_{r,s} = 0 \quad (3)$$

/where

where $\phi_{r,s}$ is the value of ϕ at the rth point in the sth row of the lattice. The scale factors in equation (2) have been chosen so that it is convenient to use a lattice for which $\delta x = \delta y = 1$ and equation (3) can then be rewritten

$$\phi_{r,s+1} + \phi_{r,s-1} + A_s(\phi_{r+1,s} + \phi_{r-1,s}) - C_s \phi_{r,s} = 0 \qquad (4)$$

For simplicity all further equations will be written for such a lattice. (Here A_s and B_s are the values of A and B applying to row s of the lattice and $C_s = 2 + 2A_s - B_s$). It may be noted that equation (4) determines a single value in row s + 1 in terms of values of ϕ in lower rows; thus

$$\phi_{r,s+1} = -\phi_{r,s-1} - A_s(\phi_{r+1,s} + \phi_{r-1,s}) + C_s \phi_{r,s} \qquad (5)$$

Although (2) is an elliptic partial differential equation, numerical solution by relaxation methods is not practical for two reasons:- namely, (a) because $B > 0$ so that if equation (4) is used to reduce the residual at point (r,s) the sum of the residuals at the surrounding points will be increased by a greater amount than the residual at (r,s) is reduced, and (b) because no boundaries can be introduced at finite distances along the x-axis without materially altering the problem (i.e., if such boundaries are introduced the solution will not satisfy the specified boundary conditions (1) and (2)).

Consequently the following method has been devised to determine the values of $\phi_{r,1}$, that is the values of ϕ on the lowest row of points above the boundary. Once these values are determined the complete field of ϕ can be determined by applying equation (5) first to determine each point of row 2 from values in row 1 and row 0 (the lower boundary), and subsequently to determine ϕ in row 3 from values in row 2 and 1 and so on.

Reference to Fig. 1 shows that application of equation (5) determines ϕ at point A in terms of the values of ϕ at C, B, D and E. Further application of the equation will express these latter values of ϕ in terms of ϕ on lower rows, so that ϕ_A can be expressed as a linear function of the values of ϕ on row 1 from point X to Y together with the values specified on the lower boundary from U to V. Similarly, the value at G, on the upper boundary, is a linear combination of the 2n + 1 values on row 1 from X to Z. (The grid is assumed to have n rows excluding upper and lower boundaries). Application of the upper boundary condition ($\phi = 0$) at each point of the upper boundary thus provides a linear equation relating 2n + 1 values of $\phi_{r,1}$.

Outside a limited range, representing the ridge over which the flow takes place (LM in Fig. 1), the boundary condition at the lower boundary is given by $\phi_{r,0} = 0$. It is easily seen therefore that the linear equation obtained by applying the upper boundary condition at all points outside the range OP has the same form, namely:

$$a_0 \phi_{r,1} + a_1 \phi_{r+1,1} + \cdots + a_{n-1} \phi_{r+n-1,1} + a_n \phi_{r+n,1} + a_{n-1} \phi_{r+n+1,1} + \cdots + a_0 \phi_{r+2n,1} = 0 \qquad (6)$$

Only 2n linearly independent solutions are possible, and we seek solutions of the form $\phi_{1,s} = D t^s$.
Equation (6) then provides an equation of order 2n, the roots of which are the possible values of t.

$$a_0 t^{2n} + a_1 t^{2n-1} + \cdots + a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 = 0 \qquad (7)$$

The method used to obtain the roots of this equation numerically is described in Appendix III.

The roots occur in pairs which are of the form
either (a) t_r and t_r^{-1} for t_r real or complex, $|t_r| \neq 1$,

/or

or (b) t_r and conjugate complex of t_r for $|t_r| = 1$.

The former type of solution increases in magnitude indefinitely either upstream or downstream, whereas the latter type (b) are oscillatory and remain finite as x tends to infinity.

Boundary condition (1) can therefore be satisfied by requiring that the solution to the left of OL (Fig. 1) should be formed of a linear combination of those solutions of type (a) corresponding values of $\phi_{r,1}$ increasing in magnitude with x increasing. Boundary condition (2) can be satisfied if the solution to the right of MP (Fig. 1) is formed of a linear combination of the remaining solutions i.e., those of type (a) decreasing with x increasing, and those of type (b) (oscillatory solutions).

It is therefore possible to define the field of ϕ in terms of $2n + m - 2$ unknowns, where m is the number of lattice columns over which the "ridge" extends in the lower boundary. These unknowns are the 2n unknown multipliers of the solutions corresponding to the 2n values of t (these solutions applying outside and on the dotted lines OL and PM in Fig. 1), together with the $m - 2$ unknown values of ϕ_1 lying between L and M. Application of the upper boundary condition at the $2n + m - 2$ points between O and P provides $2n + m - 2$ linear equations for these unknowns.

In order to simplify the algebra, the present calculations have been made in the first instance for a "ridge" represented by a non-zero value of $\phi_{r,0}$ at only a single point of the lower boundary. In this case there are 2n unknowns, $n - 1$ equations are given by the upper boundary condition and one is given the fact that ϕ_1 at the point immediately above the "ridge" must be compatible with both the upstream and downstream solutions. The solution for a ridge with any arbitrary profile is easily formed from the combination of solutions appropriate to a series of "ridges" each consisting of a displacement at single point. This is possible because the basic equation (2) is linear.

More complete details of the mathematical procedure are contained in Appendix III.

Although the solution determined by the method outlined is unique, difficulties arise in the computation because of the very high accuracy which must be maintained in certain stages of the calculation, if a satisfactory accuracy is to be achieved in the final answer. There are two main reasons for this:-

(a) The coefficients in equations (6) appropriate to a natural air-stream are large. Consequently, if when the solution at all lattice points is built up from the values of $\phi_{r,1}$, the upper boundary condition $\phi_{n+1} = 0$ is to be satisfied with an error not greater than ϵ, the values of $\phi_{r,1}$ must be known with an error not exceeding

$$\epsilon \left\{ |a_n| + 2|a_{n-1}| + 2|a_{n-2}| + \cdots + 2|a_0| \right\}^{-1}$$

In a typical case with $n = 7$ this requires an error in $\phi_{r,1}$ of the order of only 3×10^{-7} times the error which can be tolerated in ϕ_{n+1}, (i.e., in the fitting of the upper boundary condition).

(b) The equations given by the upper boundary condition for the 2n unknown coefficients of the fundamental solutions appear to form an "ill-conditioned" set. Thus the accuracy achieved in the solutions of these equations is much less than that of the calculated coefficients.

5. Results

Calculations by the method outlined have been carried out on the electronic computer at Manchester University. A programme was written by Miss M. K. Hinds to calculate the flow over a symmetrical ridge with an otherwise arbitrary profile. It was intended to use a lattice with up to 10 rows intermediate between the boundaries, but for the reasons indicated in the preceding section,

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it was found that insufficient accuracy was being achieved when the lattice included more than 6 intermediate rows.

The calculations have been carried out for a number of different sets of values of A and B representing the distribution of potential temperature and wind in typical air streams. One of these examples (Fig. 8) was made to represent the temperature and wind sounding on an occasion when gravity waves were known to have been widespread. The others were devised to illustrate the effect of various variations of stability with height, and for convenience the wind velocity in these cases was assumed to be the same at all levels.

Six intermediate rows were used in the lattice and in all cases the boundary condition $\phi_7 = 0$ was applied at the level equivalent to a pressure of 300 mb. The lower boundary of the undisturbed stream was assumed to be at 1000 mb.

The results are displayed in Figs. 2 to 10. In these diagrams the vertical displacements are plotted for stream lines at 100 mb intervals, together with the assumed ground profile. The displacements have been calculated from the relation

$$\Delta z = \frac{U_{10}\, l}{U \rho_{10}}\, \phi$$

where U_{10} and ρ_{10} are the velocity and density at 1000 mb. (No attempt has been made to space the individual stream lines in Figs. 2 to 10 at intervals appropriate to a linear scale of height).

The distribution of stability is indicated with each diagram by a tephigram on which the assumed vertical distribution of temperature is plotted. The assumed wind distribution is also given. For convenience in comparison with results given by Scorer (1954) the variation of $g^2\rho^2 B^2$ with pressure is also given. $g^2\rho^2 B^2$ is equivalent to Scorer's parameter l^2 defined by

$$l^2 = \frac{g\beta}{U^2} - \frac{\partial^2 U/\partial z^2}{U}$$

In several papers Scorer has drawn attention to the lee-waves which can form if the vertical distribution of temperature and wind is such that the parameter l^2 takes a larger value in the lower troposphere than in the upper troposphere. It is characteristic of such waves that they have their greatest amplitude in the lower region of large l and that they decrease in amplitude rapidly with height at higher levels; their wavelength L is such that $2\pi/L$ lies between the maximum and minimum values of l. Corby and Sawyer (1956) have shown that the calculated amplitude and wavelengths of such waves are very little affected by the assumption of a rigid upper boundary. "Scorer waves" of this type appear in Figs. 5, 6, 7, 8 and 10, and in Figs. 7, 8 and 10 they dominate the flow pattern to the lee of the ridge in the layers near the ground.

As indicated in section 3, waves can also appear in the solution as a direct result of the upper boundary condition. Corresponding waves can be expected to appear in the atmosphere only in so far as the tropopause acts as a rigid lid. This requires $2\pi/L$ much less than l in the stratosphere where L is the wavelength of the wave. It is characteristic of these wave solutions that they maintain their amplitude through much of the troposphere, and often tend to have greater amplitude in the upper rather than the lower troposphere. They may have one or more nodal surface and their wavelength is such that $2\pi/L$ is less than l through most of the troposphere. Their wavelength is therefore greater than any "Scorer waves" which may be present.

Such "tropopause waves" are present in Figs. 2, 3, 5, 6, 7, 9 and 10. The possible number of such waves increases as B, or l, increases (i.e., as the stability increases or the wind speed decreases). In Fig. 4 no lee-waves are

/possible

possible because the wind speed is too high, leading to a value of B which is too small for waves. Slight reduction in the wind speed with no change in the vertical stability introduces one lee-wave of long wavelength (Fig. 3). Further reduction of wind speed or increase of stability introduces a second wave with one nodal surface, and if continued a third wave with two nodal surfaces and so on. Fig. 2 which illustrates the same stability profile as Figs. 3 and 4 but a much lower wind speed contains three superposed waves.

In Figs. 2, 3 and 4 there are no large variations of l^2 with height and only "tropopause type" waves are present. On the other hand in Figs. 5, 6, 7 and 8 there is a substantial decrease of l^2 with height and "Scorer waves" are present. In Figs. 5, 6 and 7 both types of wave solution are present but in case 8 the "Scorer wave" occurs alone.

Figs. 9 and 10 are interesting as illustrating more complicated l^2 profiles. Both have an elevated layer of high stability, but that in Fig. 9 is more shallow than that of Fig. 10. The calculated stream lines, both of which contain waves, show how critically the amplitude of the calculated flow pattern may depend on the detail of the l^2 profile. This has already been pointed out by Corby and Wallington (1955).

Comment and conclusion

In interpreting the results of the present calculations in terms of atmospheric motion the restrictive assumptions must be recalled; in particular the restriction to small amplitude disturbances, and the conditions applied at the upper and lower boundaries. Formation of a turbulent layer near the ground and the formation of standing eddies is likely to make the effective profile of the ground smoother than it is in fact, and the calculated disturbance at higher levels is probably greater than that actually occurring.

The most artificial assumption is, no doubt, that of a rigid upper boundary, but it seems probable (Appendix II) that the tropopause will act in approximately the same fashion except for very short wavelength disturbances. Thus we may reasonably expect to find natural lee-waves with the same general form as those illustrated in Figs. 2 to 10.

The results illustrated in Figs. 2 to 10 can therefore be regarded as suggestive of the naturally occurring flow patterns.

Some of the features which may be expected to have counterparts in the atmosphere are:-

(a) Small values of Scorer's l throughout the atmosphere:- absence of lee-waves and approximately symmetrical hill disturbance. Disturbance decreases with height and may reverse in sign.

(b) Larger values of l (exceeding about 0.6 Km^{-1}) but without great variation with height:- lee-waves may occur owing to the effect of the tropopause; these have maximum amplitude in middle or upper troposphere; disturbance near the hill is slightly asymmetric with downward motion above the ridge line and intensified downward flow on the lee side.

(c) Large values of l in lower troposphere with small values aloft:- "Scorer" lee-waves occur; these have maximum amplitude in lower troposphere; flow over the ridge is markedly asymmetric with downward flow over the ridge line and intensified downdraft in lee of hill; "Scorer" waves may occur together with waves resulting from the tropopause and interference between wave systems may result in locally intensified up and down currents in mid-troposphere.

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Appendix I - Derivation of a differential equation for the stream lines in two-dimensional motion over a ridge using pressure as the vertical coordinate

The problem is treated as one of steady, adiabatic, inviscid flow in the x, z plane. The following notation is used:-

x, z horizontal and vertical coordinates
u, w velocities in the x and z directions
p, ρ, θ pressure, density and potential temperature
$h(p)$ height of the isobaric surface (pressure = p)
$\tilde{\omega} \equiv dp/dt$

The equations of motion in the x and z directions are:-

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (8)$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (9)$$

In most meteorological problems it is possible to assume that the atmosphere is in hydrostatic equilibrium. This is not possible in studying vertical air motions with a horizontal scale of a few kilometres, because the vertical acceleration of the air may be comparable in magnitude with its horizontal acceleration and an important contribution to the motion will arise from the oscillations of the stably stratified air column under gravity. However, it remains true in the motions which we wish to study that the pressure at any point is very close to that which would be calculated on a hydrostatic basis, and although the differences from hydrostatic pressure may not be neglected in determining the small horizontal pressure differences involved in the value of $\partial p/\partial x$ in equation (8), nevertheless it seems justifiable to use the hydrostatic approximation except in the elimination of p from equation (8). This elimination will be carried out using the exact equation (9).

If we allow the hydrostatic approximation and use isobaric coordinates, it becomes possible to express the motion in terms of a stream function, ψ, because the equation of continuity has the form,

$$\left(\frac{\partial u}{\partial x}\right)_p + \frac{\partial \tilde{\omega}}{\partial p} = 0 \qquad (10)$$

Thus we write

$$u = \frac{\partial \psi}{\partial p}, \qquad \tilde{\omega} = -\frac{\partial \psi}{\partial x} \qquad (11)$$

Since the motion is treated as steady, the potential temperature is constant along a stream line and θ is a function of ψ only. Moreover from the gas equation

$$\rho = \frac{p^{\frac{1}{\gamma}}\, p_0^{\frac{\gamma-1}{\gamma}}}{\theta R} \qquad (12)$$

where $p_0 = 1000$ mb.

The vertical velocity, w, and $\tilde{\omega}$ are related by the following equation

$$w = \frac{dz}{dt} = u\left(\frac{\partial h}{\partial x}\right)_p + \tilde{\omega}\frac{\partial z}{\partial p} \qquad (13)$$

/However

However, we shall ignore the slope of the isobaric surfaces and use the hydrostatic relation here, so that

$$w = -\tilde{\omega}/g\rho \tag{14}$$

If the left hand sides of equations (8) and (9) are rewritten in isobaric coordinates and substitution for w made from (14) we have

$$u\left(\frac{\partial u}{\partial x}\right)_p + \tilde{\omega}\left(\frac{\partial u}{\partial p}\right) = -\frac{1}{\rho}\frac{\partial h}{\partial x} = -g\left(\frac{\partial h}{\partial x}\right)_p \tag{15}$$

and

$$-\frac{1}{g}\left\{u\frac{\partial}{\partial x}\left(\frac{\tilde{\omega}}{\rho}\right) + \tilde{\omega}\frac{\partial}{\partial p}\left(\frac{\tilde{\omega}}{\rho}\right)\right\} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{16}$$

From (12)

$$\frac{1}{\rho}\frac{\partial \rho}{\partial p} = \frac{1}{\gamma p} - \frac{1}{\theta}\frac{\partial \theta}{\partial \psi}\frac{\partial \psi}{\partial p} \tag{17}$$

Thus from (16)

$$\frac{\partial h}{\partial p} = \frac{1}{-g\rho + \frac{1}{g}\left\{u\frac{\partial \tilde{\omega}}{\partial x} + \tilde{\omega}\frac{\partial \tilde{\omega}}{\partial p} - \frac{\tilde{\omega}^2}{\gamma p} + \frac{\tilde{\omega}^2}{\theta}\frac{\partial \theta}{\partial \psi}\frac{\partial \psi}{\partial p}\right\}}$$

(In evaluating the small terms involving $\frac{\partial}{\partial x}\left(\frac{\tilde{\omega}}{\rho}\right)$ the horizontal variations of density have been ignored).

Differentiation of (15) with respect to p and (17) with respect to x, and elimination of $\partial^2 h/\partial x\,\partial p$ leads to

$$u\frac{\partial^2 u}{\partial x \partial p} + \frac{\partial u}{\partial p}\frac{\partial u}{\partial x} + \tilde{\omega}\frac{\partial^2 u}{\partial p^2} + \frac{\partial \tilde{\omega}}{\partial p}\frac{\partial u}{\partial p} =$$

$$\frac{g\left\{-g\frac{\partial \rho}{\partial x} + \frac{1}{g}\frac{\partial}{\partial x}\left(u\frac{\partial \tilde{\omega}}{\partial x} + \tilde{\omega}\frac{\partial \tilde{\omega}}{\partial p} - \frac{\tilde{\omega}}{\gamma p} + \frac{\tilde{\omega}^2}{\theta}\frac{\partial \theta}{\partial \psi}\frac{\partial \psi}{\partial p}\right)\right\}}{\left\{-g\rho + \frac{1}{g}\left(u\frac{\partial \tilde{\omega}}{\partial x} + \tilde{\omega}\frac{\partial \tilde{\omega}}{\partial p} - \frac{\tilde{\omega}^2}{\gamma p} + \frac{\tilde{\omega}^2}{\theta}\frac{\partial \theta}{\partial \psi}\frac{\partial \psi}{\partial p}\right\}^2}$$

We may neglect all terms except $g^2\rho^2$ in the denominator - this is equivalent to making the hydrostatic approximation. Moreover if the vertical scale of the disturbances is small compared with the height of the 300 mb surface the last two terms in the bracket in the numerator are also smaller than the immediately preceding term which will itself be neglected when conditions appropriate to small perturbations are introduced. Making these approximations and substituting from (11) we have

$$\frac{\partial \psi}{\partial p}\frac{\partial^3 \psi}{\partial x \partial p^2} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial p^3} = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial x} - \frac{1}{\rho^2 g^2}\left\{\frac{\partial \psi}{\partial p}\frac{\partial^3 \psi}{\partial x^3} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial x^2 \partial p}\right\} \tag{18}$$

Substituting from (12)

$$\frac{\partial \psi}{\partial p}\frac{\partial^3 \psi}{\partial x \partial p^2} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial p^3} = \frac{R}{p^{\frac{\gamma-1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}}}\frac{\partial \theta}{\partial \psi}\frac{\partial \psi}{\partial x} - \left\{\frac{R\theta}{g p^{\frac{\gamma-1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}}}\right\}^2\left\{\frac{\partial \psi}{\partial p}\frac{\partial^3 \psi}{\partial x^3} - \frac{\partial \psi}{\partial x}\frac{\partial^3 \psi}{\partial x^2 \partial p}\right\} \tag{19}$$

If we now restrict our study to small disturbances of a uniform flow, write $\psi = \psi_0 + \phi$ and neglect terms of the second order in ϕ

/equation

equation (19) becomes

$$\frac{\partial \psi_0}{\partial p}\frac{\partial^3 \phi}{\partial x\,\partial p^2} - \frac{\partial^3 \psi_0}{\partial p^3}\frac{\partial \phi}{\partial x} = \frac{R}{p^{\frac{1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}}}\frac{\partial \theta}{\partial \psi_0}\frac{\partial \phi}{\partial x} - \left\{\frac{R\theta}{g p^{\frac{1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}}}\right\}\left\{\frac{\partial \psi_0}{\partial p}\frac{\partial^3 \phi}{\partial x^3}\right\} \quad (20)$$

This equation may be integrated with respect to x, so that making use of the fact that $\phi \to 0$ as $x \to -\infty$ i.e., upstream in the undisturbed stream, we have

$$\frac{\partial^2 \phi}{\partial p^2} + A\frac{\partial^2 \phi}{\partial x^2} + B\phi = 0$$

where $$A = \left\{\frac{R\theta}{g p^{\frac{1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}}}\right\}^2$$

$$B = -\left\{\frac{R}{U^2 p^{\frac{1}{\gamma}} p_0^{\frac{\gamma-1}{\gamma}}}\left[\frac{\partial \theta}{\partial p}\right]_0 + \frac{\partial^2 U/\partial p^2}{U}\right\}$$

and $U = \left[\frac{\partial \psi}{\partial p}\right]_0$ = velocity in the undisturbed stream.

For the purposes of numerical solution, it is convenient to use units for x and p comparable with the mesh length of the grid upon which (21) is to be solved. A convenient choice of unit for x is 1 Km and for p_0, 100 mb. With these units the expressions for A and B become

$$A = 2{\cdot}2811 \times 10^{-4}\ \theta^2 p^{-\frac{1}{\gamma}}$$

$$B = -\left\{\frac{148{\cdot}16}{U^2 p^{\frac{1}{\gamma}}}\left[\frac{\partial \theta}{\partial p}\right]_0 + \frac{\partial^2 U/\partial p^2}{U}\right\}$$

where θ is expressed in $^{\circ}$A and U in m/s.

Appendix II - Lee-waves arising from the presence of a stable stratosphere

It has been mentioned that solutions of equation (1) or (2) based on the presence of a rigid upper boundary can contain oscillations corresponding to lee-waves. In the atmosphere no rigid upper boundary exists, but a marked change of stability is usually present at the tropopause, and it is the purpose of this appendix to show that the existence of a stable stratosphere can lead to wave motions very similar to those arising from the imposition of a rigid upper boundary.

It is convenient to return to Scorer's formulation of the problem, and to consider the amplitude, ζ , of a simple harmonic oscillation of the form $w = \zeta e^{ikx}$. Substitution in equation (1) leads to

$$\frac{\partial^2 \zeta^2}{\partial z^2} + (l^2 - k^2)\zeta = 0 \qquad (22)$$

where $$l^2 = \frac{g}{U^2\theta}\frac{\partial \theta}{\partial z} - \frac{1}{U}\frac{\partial^2 U}{\partial z^2}$$

If the shape of the ridge over which the flow is required is represented by a Fourier integral

$$H(x) = \text{real part of } \int_0^\infty \zeta_0(k) e^{ikx}\, dk$$

then the lower boundary condition for the solution of (22) is

$$\zeta = \zeta_0 \text{ at } z = 0.$$

We seek solutions of the form

$$\zeta = \int_0^\infty \zeta(k) e^{ikx}\, dk \qquad (23)$$

In order to compare the effect of a stable stratosphere with that of a rigid lid, it is convenient to consider the simplified problem in which the stability parameter l is constant in both troposphere and stratosphere. We may then compare the solution for this case with a single layer representation in which a rigid lid is imposed at the tropopause. The two models of the atmosphere which are to be compared are then:-

Rigid lid: $l = l_1$, boundary condition $\zeta = 0$ at $z = a$

Two-layer $l = l_1$ in the lower layer, $z < a$
$l = l_2$ in the upper layer, $z > a$
$l_2 > l_1$.

Following Corby and Sawyer (1956) an appropriate upper boundary condition in the two-layer model is obtained by requiring that in the upper layer $\zeta(k)$ should have the form $\zeta(k) = A e^{i\nu_2 z}$ where $\nu_2^2 = l_2^2 - k^2$.
In the case of the rigid lid it is easily shown that

$$\zeta(k) = \zeta_0(k)\left\{\cos \nu_1 z - \frac{1}{t_1}\sin \nu_1 z\right\} \qquad (24)$$

where $t_1 = \tan a\nu_1$ and $\nu_1^2 = l_1^2 - k^2$.
However when an attempt is made to reconstruct the flow by carrying out the integration $\xi = \int_0^\infty \zeta(k) e^{ikx}\, dk$ difficulty arises because the integral is

/indeterminate

indeterminate around the singularities of $\zeta(k)$ which arise where $\tan a\nu_1 = 0$. This difficulty is resolved by Lamb (1916) in a similar problem by introducing internal friction into the controlling equation, and thereby displacing the poles of $\zeta(k)$ slightly off the real axis in the complex k-plane towards the positive imaginary side.

Scorer (1949) adopts a solution which is equivalent to Lamb's with the result that the integral $\int_0^\infty \zeta(k) e^{ikx}\, dk$ can be evaluated by integrating in the complex k-plane along a path following the real axis but passing just on the negative imaginary side of any poles of $\zeta(k)$, (Fig.11(a)). The integral may be approximated by completing the contour by means of an arc of the circle at infinity and a convenient radius. For $x > 0$ the contour must be completed on the positive imaginary side of the real axis and for $x < 0$ on the negative imaginary side so that the factor $e^{ikx} \to 0$ on the circle at infinity in both cases. As a result the solutions for positive and negative x differ by an amount given approximately by $2\pi i \times$ (sum of the residues at the poles of $\zeta(k)$.

This part of the solution is identified as the lee-waves and appears as an infinite train of harmonic oscillations on the downstream side of the ridge ($x > 0$). The remaining part of the solution is approximately symmetrical about $x = 0$ and decreases to zero as x increases.

In the particular case of a stream with constant l flowing beneath a rigid lid, the lee-wave part of the solution is $2\pi i \times$ (residue of

$$\zeta_0(k)\left\{\cos \nu_1 z - \frac{1}{t_1}\sin \nu_1 z\right\} e^{ikx} \text{ at its poles)}$$
$$= -\sum \frac{2\pi \zeta_0 \nu_1}{a k_0} \sin \nu_1 z (\sin k_0 x - i \cos k_0 x)$$

(the summation being over all poles).

We are interested only in the real part of the solution which represents a lee-wave train with amplitude $\frac{2\pi \zeta_0 \nu_1 \sin \nu_1 z}{a k_0}$ corresponding to each k_0 which satisfies $\tan a\nu_1 = 0$.

Turning now to the two-layer model, conditions of continuity at the interface lead to an equation corresponding to (24), namely:-

$$\zeta(k) = \zeta_0(k)\left\{\cos \nu_1 z + \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1}\sin \nu_1 z\right\} \qquad (25)$$

The lee-waves will be discussed as though only one set of lee-waves were present in the corresponding model with a lid (i.e., $\tan a\nu_1 = 0$ had only one solution), but the treatment is essentially the same if more than one lee-wavelength is possible.

The right hand side of equation (25) has no poles on the real axis, but if $\nu_2 \gg \nu_1$, for each pole in the single layer case with a lid there is a corresponding pole at a neighbouring position for which $t_1 = -i\nu_1/\nu_2$ (approx.).

If k_0 is the value of k which results in lee-waves in the "lid-model", the corresponding value of $t_1 = 0$.

Moreover for small increments,

$$\delta t_1 = a \sec^2 \nu_1 \delta\nu_1 = -\frac{a \sec^2 \nu_1 \, k \delta k}{\nu_1} \qquad (26)$$

Thus, since ν_1/ν_2 is supposed to be small, the value of k corresponding to the pole in the two-layer model is approximately

$$k_0 + (i\nu_1^2/a\nu_2 k_0)$$

where k_0 is the value of k for the pole in the model with a lid.

In the neighbourhood of this pole the expression (25) becomes large, and there is thus a small range of k with width of the order ν_1/ν_2 for which the integrand in (23) is large, and makes a contribution to the integral comparable with that from the remainder of the range of integration. The contribution from the small range of integration near the pole corresponds to the lee-wave component in the model with a lid which arises in a similar way. This can be made clearer by an examination of the integral by contour integration methods.

The integral (23) has to be evaluated along OX in the complex k-plane. In order to examine the integral over a small section A B from $k = k_0 - \epsilon$ to $k = k_0 + \epsilon$, the line A B is replaced by the line A C parallel to the imaginary axis, the line C D at infinity and D B also parallel to the imaginary axis (see Fig.11(b)). If C D is chosen on the positive side of the real axis for $x > 0$ and on the negative side for $x < 0$ (e.g., C'D' in Fig.11(b)), then the factor e^{ikx} introduces a factor into the integrand which decreases exponentially with the imaginary part of k. The contribution to the integral from C D or C'D' can therefore be neglected.

If we write $k = k_1 + i\,k_2$, $k_1 = k_0 - \epsilon$ on A C and $k_1 = k_0 + \epsilon$ on D B the contribution from the integrals along A C and D B can be evaluated as follows

$$\int_A^C \xi(k)\, e^{ikx}\, dk - \int_B^D \xi(k)\, e^{ikx}\, dk$$

$$= \int_0^\infty dk_2 \left\{ \left[\xi_0(k) \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1} \sin \nu_1 z\, e^{ikz} \right]_{k_0-\epsilon+ik_2} - \left[\xi_0(k) \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1} \sin \nu_1 z\, e^{ikx} \right]_{k_0+\epsilon+ik_2} \right\}$$

$$= 2\epsilon \int_0^\infty dk_2 \frac{\partial}{\partial k_1} \left\{ \xi_0(k) \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1} \sin \nu_1 z\, e^{ikx} \right\}$$

$$= 2\epsilon \int_0^\infty \left[\frac{\partial}{\partial k_1} \left\{ \xi_0(k) \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1} \sin \nu_1 z \right\} e^{ik_1 x} + ix\, \xi_0(k) \frac{\nu_1 t_1 + i\nu_2}{\nu_1 - i\nu_2 t_1} \sin \nu_1 z\, e^{ik_1 x} \right] e^{-k_2 x}\, dk_2$$

Since ϵ can be chosen so that $|\nu_1 - i\nu_2 t|$ does not approach zero on A C or D B the preceding integral is less than $2\epsilon M e^{-k_2 x}/x$ where M is the upper bound of the modulus of the expression enclosed by square brackets within the integral, showing that the integral along A C D B makes a contribution to the integral which varies with ϵ and is of the same order of magnitude as the contribution from other parts of the range of integration outside A B.

Thus we have:-

for $x > 0$ $$\int_0^x \xi(k)\, e^{ikx}\, dk = \int_{OACDBX} \xi(k)\, e^{ikx}\, dk + 2\pi i \text{ (residue at pole)} \quad (27)$$

for $x < 0$ $$\int_0^x \xi(k)\, e^{ikx}\, dk = \int_{OAC'D'BX} \xi(k)\, e^{ikx}\, dk \quad (28)$$

In the first integral on the right hand side of these equations no special contribution arises from any part of the range of integration and since the contribution from the section A B is essentially proportional to ϵ, it is small compared with the remainder of the integral.

The second term on the right of (27) can be identified with the lee-wave component. As a result of the adoption of solutions of the form $\xi(k) = A e^{i\nu_2 z}$ in the upper layer, the lee-waves occur only on the downstream side of the ridge.

In evaluating the residue at the pole we recall that

$$\frac{\partial}{\partial k}(\nu_1 - i\nu_2 t) = -\frac{k}{\nu_1} + i\frac{kt_1}{\nu_2} + i\frac{\nu_2 a k}{\nu_1}$$

$$= \frac{i\nu_2 a k}{\nu_1} \quad \text{(approx.) since } \nu_2/\nu_1 \text{ is large.}$$

Thus, the lee-wave component in (27) has the approximate form

$$\frac{2\pi\xi_0\nu_1}{ak_0}\sin\nu_1 z\, e^{ik_0 x}\, e^{-\frac{x\nu_1^2}{a\nu_2 k_0}}$$

This is the same as the form obtained in the model with the rigid lid except for the factor $e^{-\frac{x\nu_1^2}{a\nu_2 k_0}}$ which indicates an exponential decrease with x. Thus in the two-layer atmosphere lee-waves occur with amplitude near the mountain and wavelength similar to those computed with the artificial boundary condition represented by the rigid lid. However the waves decay downstream and are reduced to about half their original amplitude after a distance $x = a\nu_2 k_0 \log_e 2$ or after a number of lee-waves equal to $\frac{\log_e 2\, a\, \nu_2 k_0^2}{2\pi\nu_1^2}$

Although the preceding theory has been developed on the assumption that l_2 is much greater than l_1, there is little reason to doubt that some similar form of lee-waves will exist when l_2 and l_1 have values typical of the stratosphere and troposphere respectively. Typical values would be $l_1^2 = 0.89$ Km^{-2} and $l_2^2 = 6.75$ Km^{-2} leading to $\nu_1 = 0.63$ Km^{-1}, $\nu_2 = 2.5$ Km^{-1} for $k = 0.7$ Km^{-1} and $a = 10$ Km. The above analysis then suggests that the amplitude of the lee-wave train falls to half its original value after three oscillations.

Appendix III - Notes on mathematical technique

1. Method of obtaining values of t corresponding to possible solutions of the form $\phi_{s,1} = Dt^s$ at a distance from the ridge

In section 4 of the present paper it has been shown that with a lower boundary condition $\phi_0 = 0$ and $\phi_{s,1} = Dt^s$ at points on the lowest internal line of the grid, the upper boundary condition $\phi_{n+1} = 0$ applied at the $n+1^{th}$ row of the grid leads to an equation for t of the form:-

$$a_0 t^{2n} + a_1 t^{2n-1} + \text{----------} + a_n t^n + a_{n-1} t^{n-1} + \text{----------} + a_1 t + a_0 = 0 \quad (7)$$

This equation has roots in pairs of the form t_r, $1/t_r$ and in order to obtain these it is convenient to transform the equation to one with x as variable, where

$$x = t + \frac{1}{t}$$

and the roots t_r, $1/t_r$ are obtained as roots of the quadratic equation

$$t^2 - xt + 1 = 0.$$

As explained in section 4 the value of $\phi_{r,s}$, the disturbance of the stream function at a point in the s^{th} internal row of the grid can be expressed as a linear function of the values $\phi_{j,1}$ in row 1 symmetrical about the point (r,1). For any solution which has the form $\phi_{j,1} = Dt^j$ corresponding values spaced equally on either side of $\phi_{r,1}$ are respectively $t^m \phi_{r,1}$ and $t^{-m} \phi_{r,1}$, so that $\phi_{r,s}$ can be expressed as $\phi_{r,1}$ multiplied by a polynomial in x of degree s - 1. A recurrence relation which enables the coefficients of the polynomial to be calculated is obtained from the finite difference form of the equation governing the flow (4), namely:-

$$\phi_{r,s+1} + \phi_{r,s-1} + A_s \left(\phi_{r+1,s} + \phi_{r-1,s} \right) - C_s \phi_{r,s} = 0 \quad (4)$$

If we suppose $\phi_{r,s} = \phi_{r,1} \left\{ \sum_{i=0}^{s-1} b_{i,s} x^i \right\}$ substitution in (4)

gives

$$\sum_{i=0}^{s} b_{i,s+1} x^i + \sum_{i=0}^{s-2} b_{i,s-1} x^i + A \left\{ t \sum_{i=0}^{s-1} b_{i,s} x^i + \frac{1}{t} \sum_{i=0}^{s-1} b_{i,s} x^i \right\} - C_s \sum_{i=0}^{s-1} b_{i,s} x^i = 0$$

Equating coefficients

$$b_{i,s+1} = C_s b_{i,s} - A_s b_{i-1,s} - b_{i,s-1} \qquad i \neq 0 \quad (29)$$

This relation also applies for i = 0 provided $b_{i,s}$ is treated as zero for negative i.

Commencing with $b_{i,0} = 0$ for all i and $b_{0,1} = 1$, $b_{i,1} = 0$ $i \neq 0$, the coefficients $b_{i,s}$ can be calculated for any row s from the recurrence relation (29). The upper boundary condition $\phi_{r,n+1} = 0$ then yields the

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equation

$$\sum_{i=0}^{n} b_{i,n+1}\, x^n = 0 \qquad (30)$$

In the numerical treatment of the problem the coefficients $b_{i,n+1}$ are first calculated by means of the recurrence relations (29). Equation (30) was then solved for x. Since allowance had to be made for the possible existence of imaginary roots, the equation was first made of even degree by multiplying by x, if necessary. The quadratic factors were then evaluated numerically. To determine each factor 10 iterations were carried out by the "penultimate remainder method" (Aitken, 1951). Subsequently the iteration was continued by "Blairstow method" (Olver, 1951), because this appeared to converge more rapidly in the later stages. Iteration was continued until the change in the coefficients was less than 10^{-6}, the equation was then divided through by the quadratic factor and a further factor sought. Each quadratic factor was tested against the full equation (30) and further iterations carried out by Blairstow method, if necessary, to maintain the specified accuracy.

The values of x and the corresponding values of t were obtained by solving for x the quadratic equations provided by each factor, and then solving the equation

$$t^2 - x\,t + 1 = 0$$

for t.

The solutions for t were sorted into those corresponding to solutions for $\phi_{r,s}$ which increase or decrease indefinitely downstream, namely those for which $|t| \neq 1$ (that is x real and $|x| > 2$ or x complex) on the one hand, and those for which $\phi_{r,s}$ oscillates but remains finite, namely those for which $|t| = 1$ (that is x real and $|x| \leq 2$) on the other. In the following paragraphs the former solutions will be denoted by t_p and the latter by t_q.

2. Determination of the field of $\phi_{r,s}$

The next stage was to determine the disturbance upstream and downstream of a ridge represented by a non-zero value of ϕ at only a single point of the lower boundary, namely the origin (0, 0). As indicated in section 4 the appropriate solutions on the lowest internal row of the grid can be represented by

$$\phi_{r,1} = \sum_p P'_p t_p^r \qquad \text{for } r < 0$$

$$\phi_{r,1} = \sum_p P_p t_p^{-r} + \sum_q \left(Q_q t_q^{-r} - Q'_q t_q^r \right) \qquad \text{for } r > 0$$

Now it has been seen (equation (6)) that the upper boundary condition applied at a single point of the boundary provides a linear equation relating $\phi_{r,1}$ at 2n +1 points of row 1.

If we write S_s for either P_p or Q_q, so that summation can be carried out over both sets of coefficients, it is easily seen that applying the upper boundary condition at the point (-n+1, n+1) and making use of the fact that t_s satisfies equation (7), leads to the equation

$$\sum_s a_0 \left(S_s t_s^{-1} - S_s' t_s \right) + \epsilon_1 = 0 \qquad (31)$$

where ϵ_1 is the value of ϕ at this boundary point if $\phi_{i,1} = 0$ for all i, $\phi_{0,0} = 1$ and $\phi_{j,0} = 0$ for $j \neq 0$.

Application of the boundary condition successively at points (-n+2, n+1) (-n+3, n+1) and so on, leads to equations

$$\left.\begin{aligned} &\sum_s a_0 \left(S_s t_s^{-2} - S_s' t_s^2 \right) + \sum_s a_1 \left(S_s t_s^{-1} - S_s' t_s \right) + \epsilon_2 = 0 \\ &\sum_s a_0 \left(S_s t_s^{-3} - S_s' t_s^3 \right) + \sum_s a_1 \left(S_s t_s^{-2} - S_s' t_s^2 \right) + \sum_s a_2 \left(S_s t_s^{-1} + S_s' t_s \right) + \epsilon_3 = 0 \\ &\text{---------} + \text{---------} + \text{-------} + \cdots \end{aligned}\right\} \qquad (32)$$

n such equations can be formed.

Application of the boundary condition at the points (n-1, n+1), (n - 2, n + 1) (n - 3, n + 1) etc. leads to a similar set of equations in which the position of S, and S′ are reversed.

Since the values of ϵ_r are symmetrical about the point (0, n + 1) subtraction of these sets of equations in pairs gives a set of n independent equations for $S_s - S_s'$ namely

$$\sum_s (S_s - S_s') a_0 \left(t_s^{-1} - t_s \right) = 0$$

$$\sum_s (S_s - S_s') \left\{ a_0 \left(t_s^{-2} - t_s^2 \right) + a_1 \left(t_s^{-1} - t_s \right) \right\} = 0$$

$$\sum_s (S_s - S_s') \left\{ a_0 \left(t_s^{-3} - t_s^3 \right) + a_1 \left(t_s^{-2} - t_s^2 \right) + a_2 \left(t_s^{-1} - t_s \right) \right\} = 0$$

etc., etc.

These can only be satisfied if $S_s = S_s'$ for all s and the preceding equations

/(32)

(32) reduce to

$$\left.\begin{aligned}
\sum_s S_s\, a_0 \left\{ t_s - t_s^{-1} \right\} &= \epsilon_1 \\
\sum_s S_s \left\{ a_0 (t_s^2 - t_s^{-2}) + a_1 (t_s - t_s^{-1}) \right\} &= \epsilon_2 \\
\sum_s S_s \left\{ a_0 (t_s^3 - t_s^{-3}) + a_1 (t_s^2 - t_s^{-2}) + a_2 (t_s - t_s^{-1}) \right\} &= \epsilon_3 \\
\text{etc.,} \qquad\qquad \text{etc.,}
\end{aligned}\right\} \quad (33)$$

For convenience in manipulation these equations can be rewritten

$$\sum_s S_s (t_s^r - t_s^{-r}) = F(r) \qquad r = 1, 2, 3 \ldots n \quad (34)$$

where F(r) is obtained from the recurrence relation

$$\left.\begin{aligned}
F(r) &= a_0^{-1} \left\{ \epsilon_r - \sum_{i=1}^{r-1} a_{r-i}\, F(i) \right\} \\
F(1) &= a_0^{-1}\, \epsilon_1
\end{aligned}\right\} \quad (35)$$

The values of $a_{r,}$ and ϵ_r required to calculate F(i) from equations (35) can themselves be derived from the finite difference equation (4). If we denote by a_{rs} and ϵ_{rs} the values of a_r and ϵ_r appropriate to an upper boundary at the Sth row of the grid then equation (4) leads directly to the recurrence formulae

$$\left.\begin{aligned}
a_{r,s+1} &= C_s\, a_{r,s} - A_s (a_{r-1,s} + a_{r+1,s}) - a_{r,s-1} \\
\epsilon_{r,s+1} &= C_s\, \epsilon_{r,s} - A_s (\epsilon_{r-1,s} + \epsilon_{r+1,s}) - \epsilon_{r,s-1}
\end{aligned}\right\} \quad (36)$$

with the starting conditions

$$\left.\begin{aligned}
a_{r,0} &= 0 \text{ for all r,} \\
a_{0,1} &= 1 \qquad a_{r,1} = 0 \qquad r \neq 0 \\
\epsilon_{0,0} &= 1 \qquad \epsilon_{r,0} = 0 \qquad r \neq 0 \\
\epsilon_{r,1} &= 0 \quad \text{for all r.}
\end{aligned}\right\} \quad (37)$$

The relations (36) and (37) were used to calculate a_r and ϵ_r and from these equations (35) were used to calculate the constant terms in equations (34). This set of n linear equations was then solved for the coefficients S_s and, using these, the solutions on the lowest row of the grid, $\phi_{r,1}$ could be obtained.

/The

The solution appropriate to a ridge represented by non-zero values of the stream-function at more than one point of the lower boundary was then derived by superposition of appropriate multiples of the solution obtained for the case of a single non-zero value. Subsequently the solution for the whole field was reconstructed using the finite difference equation (4).

A check on the accuracy of the procedure should have been provided by the vanishing of $\phi_{r,n+1}$ on the upper boundary. However as noted in section 4 of the paper a very high degree of accuracy is needed in intermediate stages of the calculation if this condition is to be satisfied. In the calculations now reported adequate accuracy could not be achieved with $n = 7$ and even with $n = 6$ no satisfactory check at the upper boundary was achieved in all cases. (The individual diagrams illustrate the stream lines computed at the upper boundary which should have been straight and horizontal.)

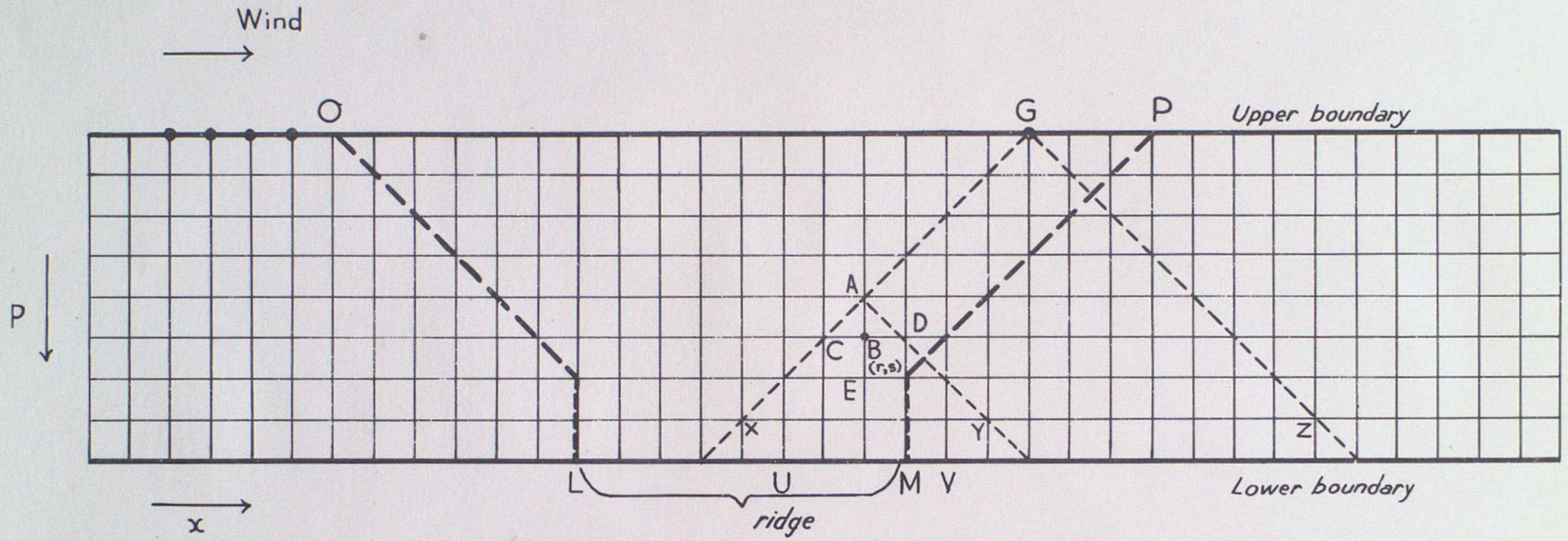
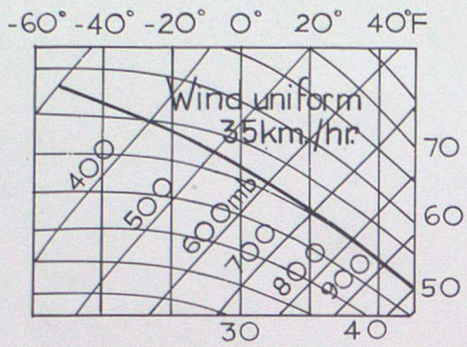
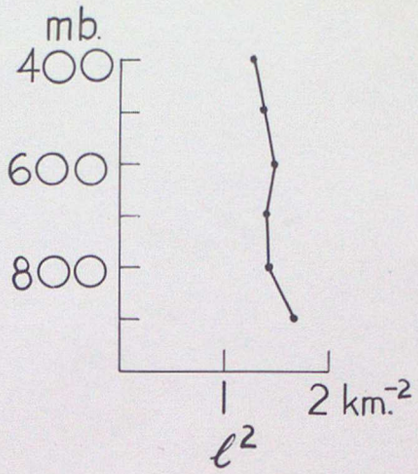
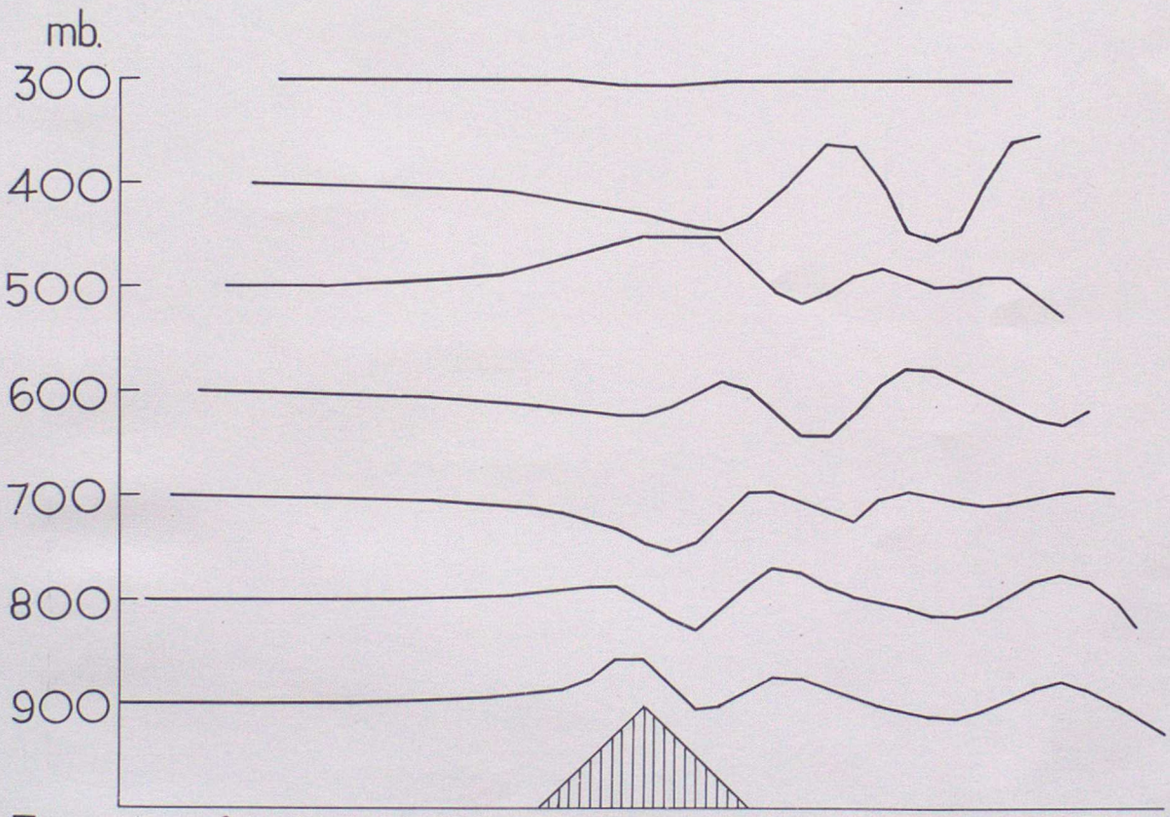

Wind
O
G
P
Upper boundary
P
A
D
C
B
(r,s)
E
X
Y
Z
L
U
M
V
ridge
Lower boundary
x

Fig. 1

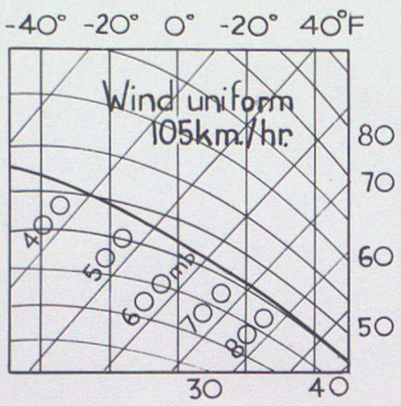


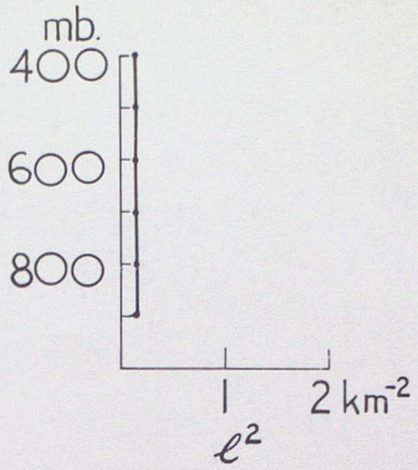


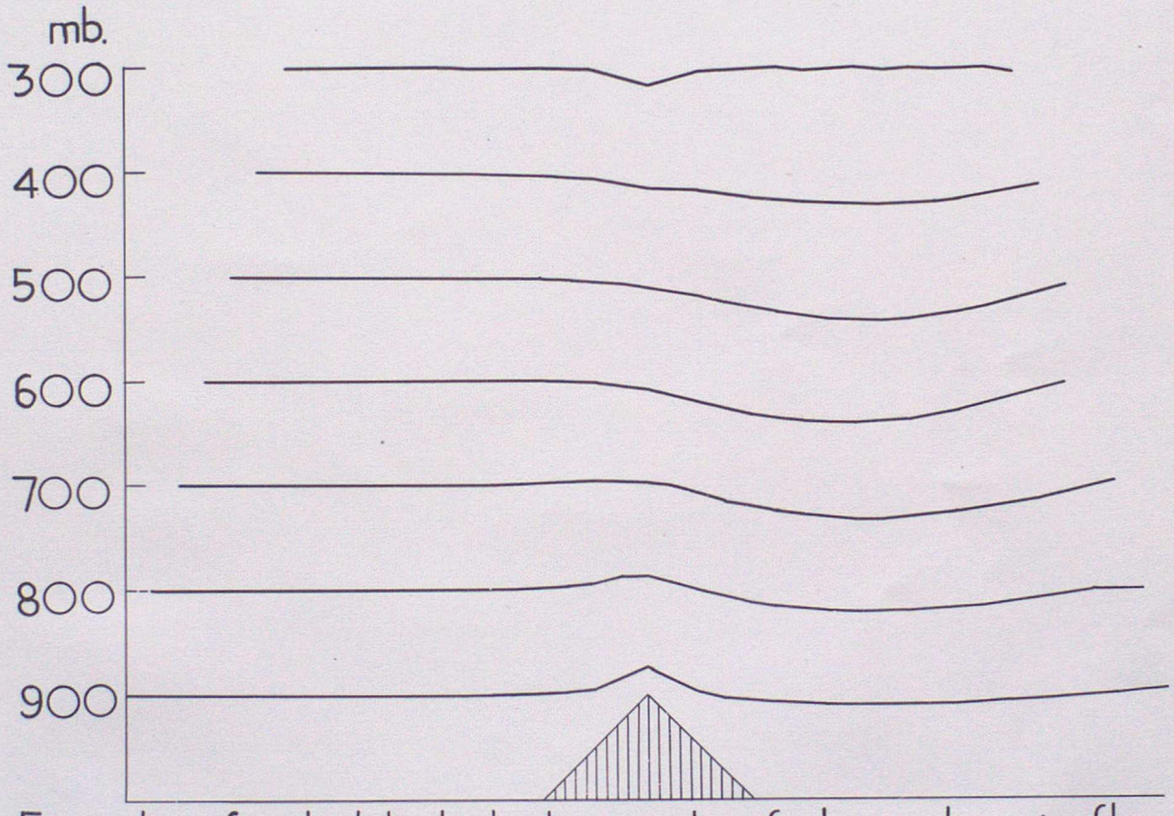


Example of calculated displacements of stream lines in flow over a ridge

Fig. 2

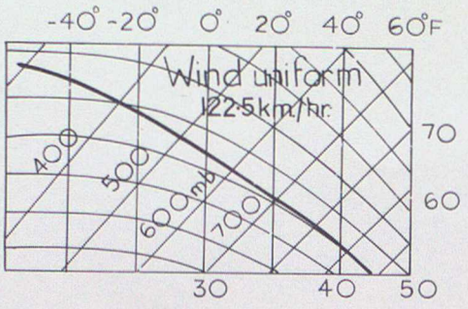


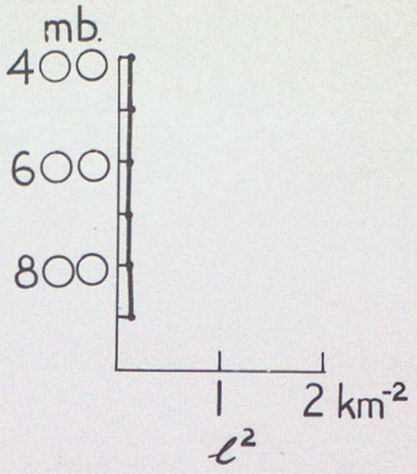


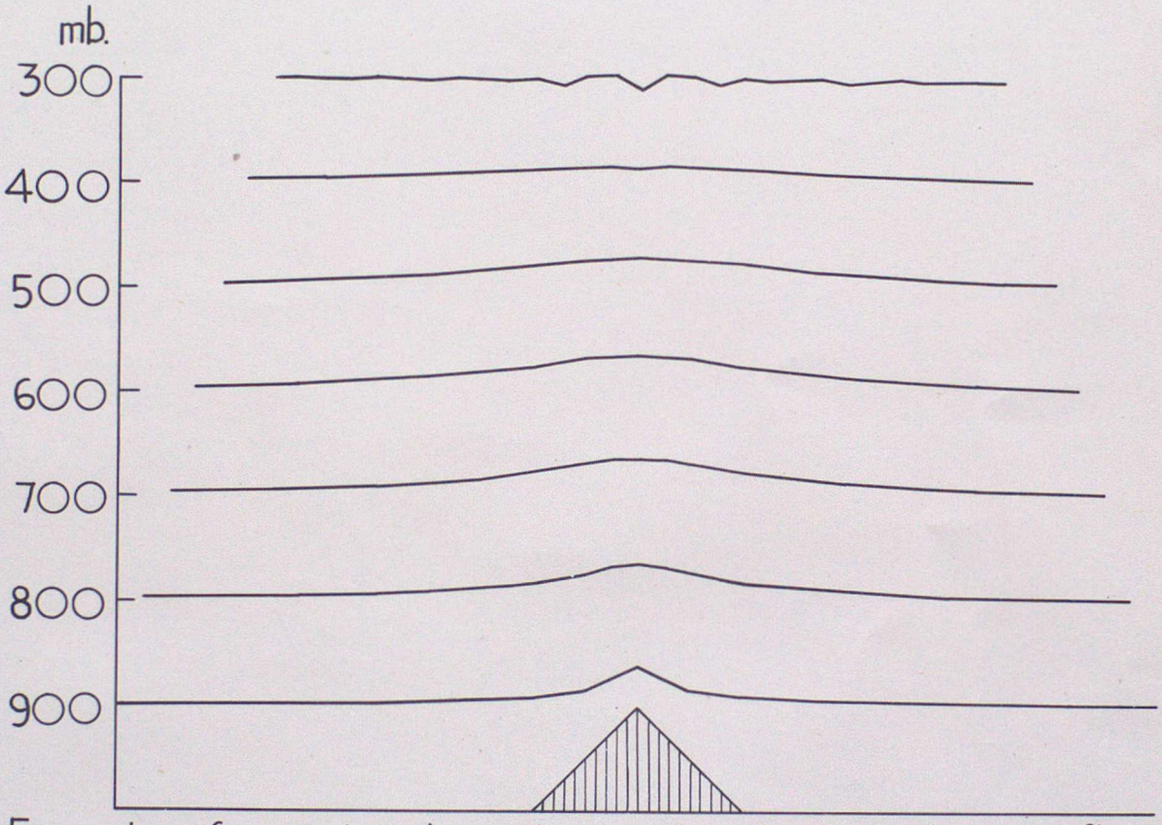


Example of calculated displacements of stream lines in flow over a ridge

Fig. 3

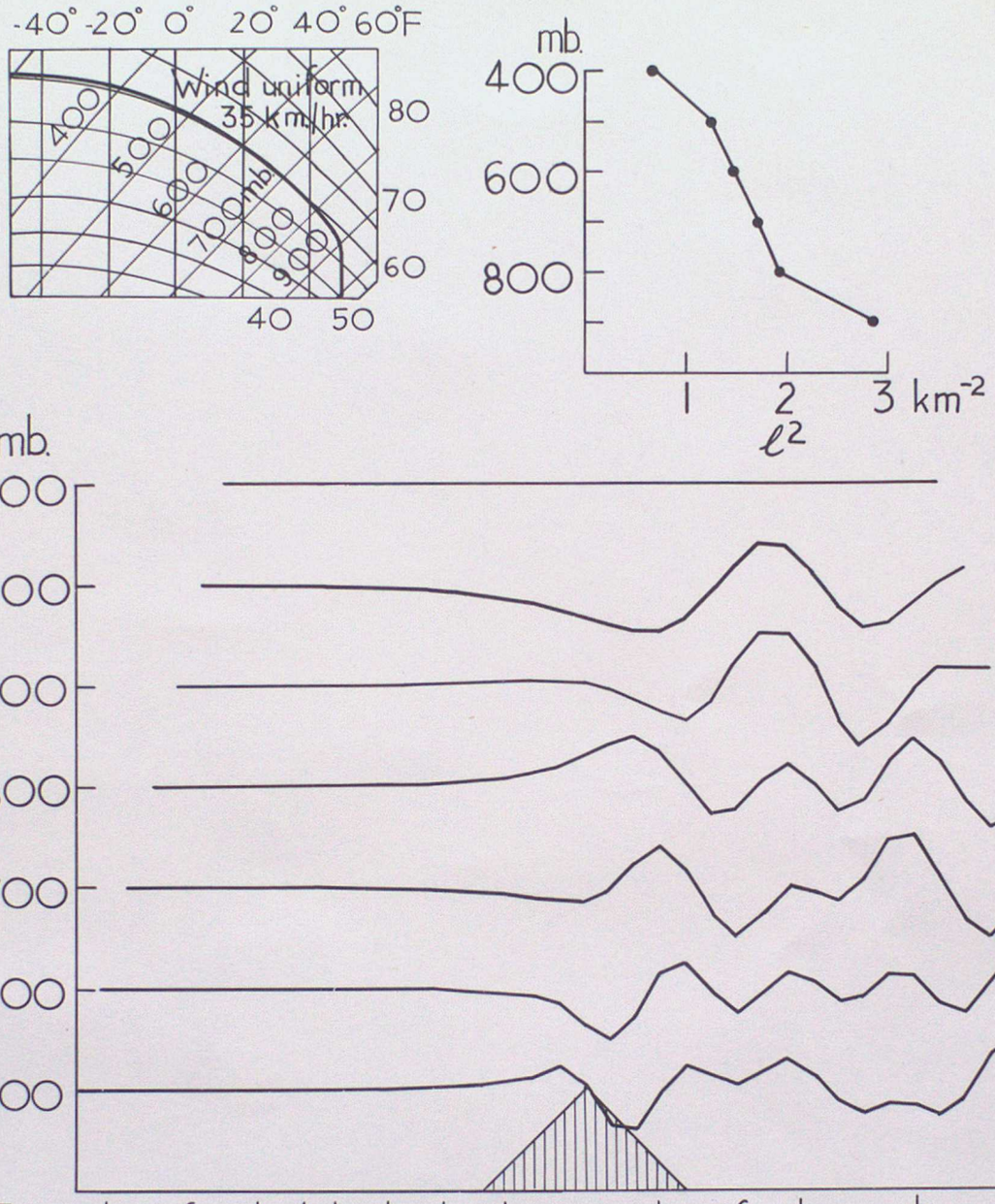






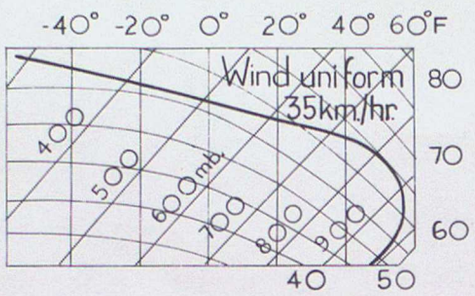
Example of calculated displacements of stream lines in flow over a ridge

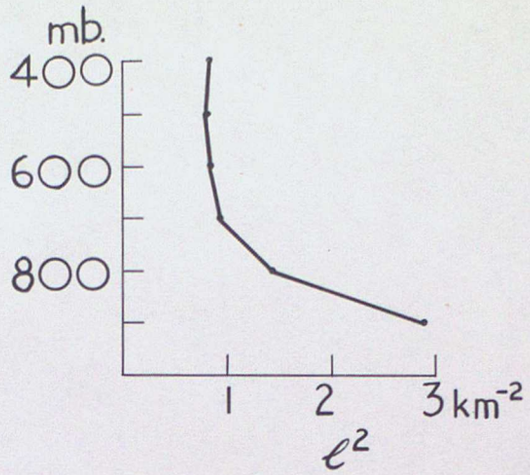
Fig. 4

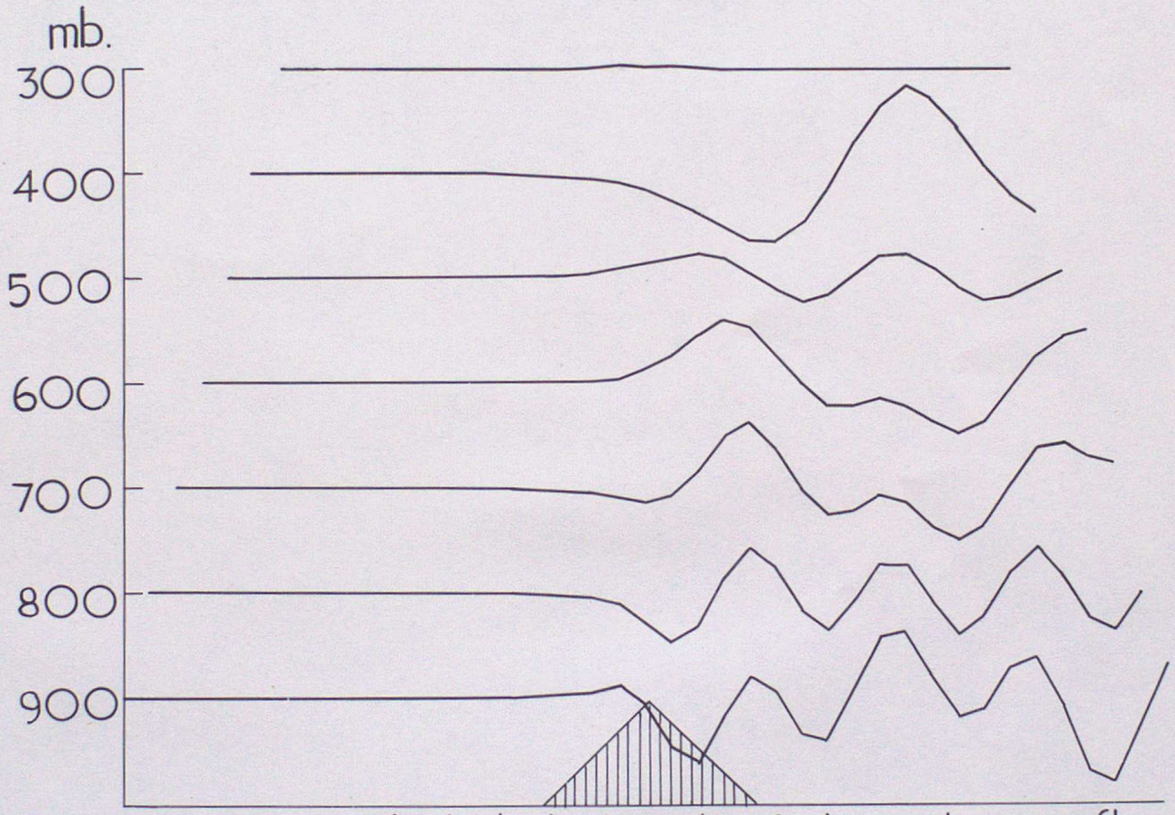


Example of calculated displacements of stream lines in flow over a ridge

Fig. 5

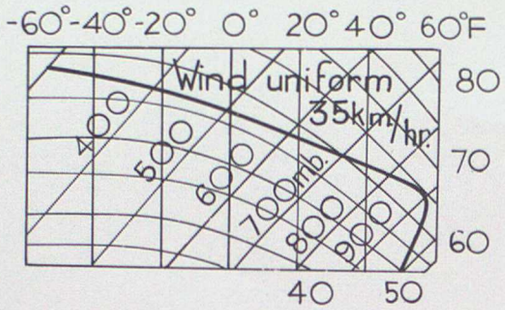


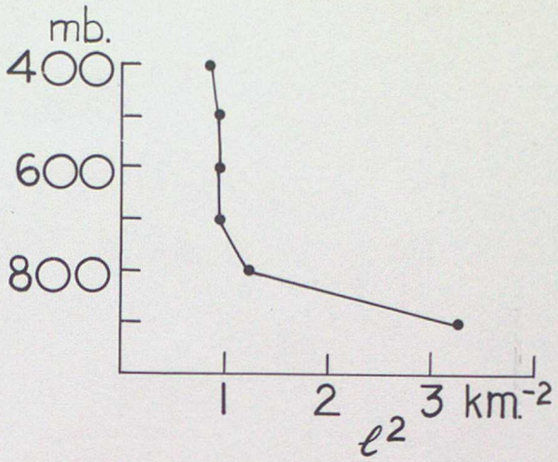


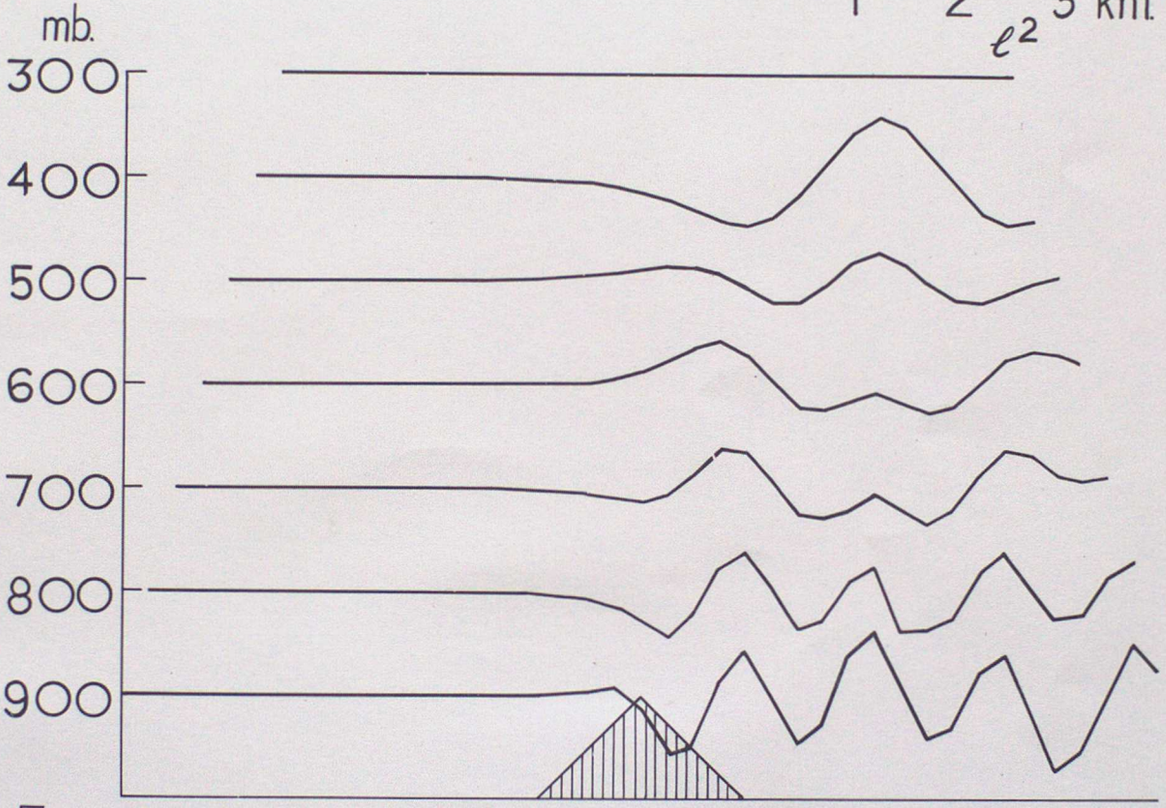


Example of calculated displacements of stream lines in flow over a ridge

Fig. 6

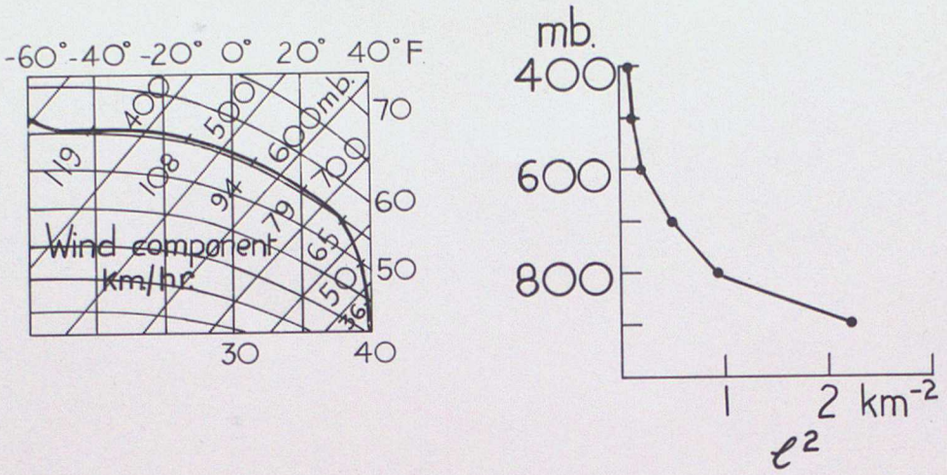


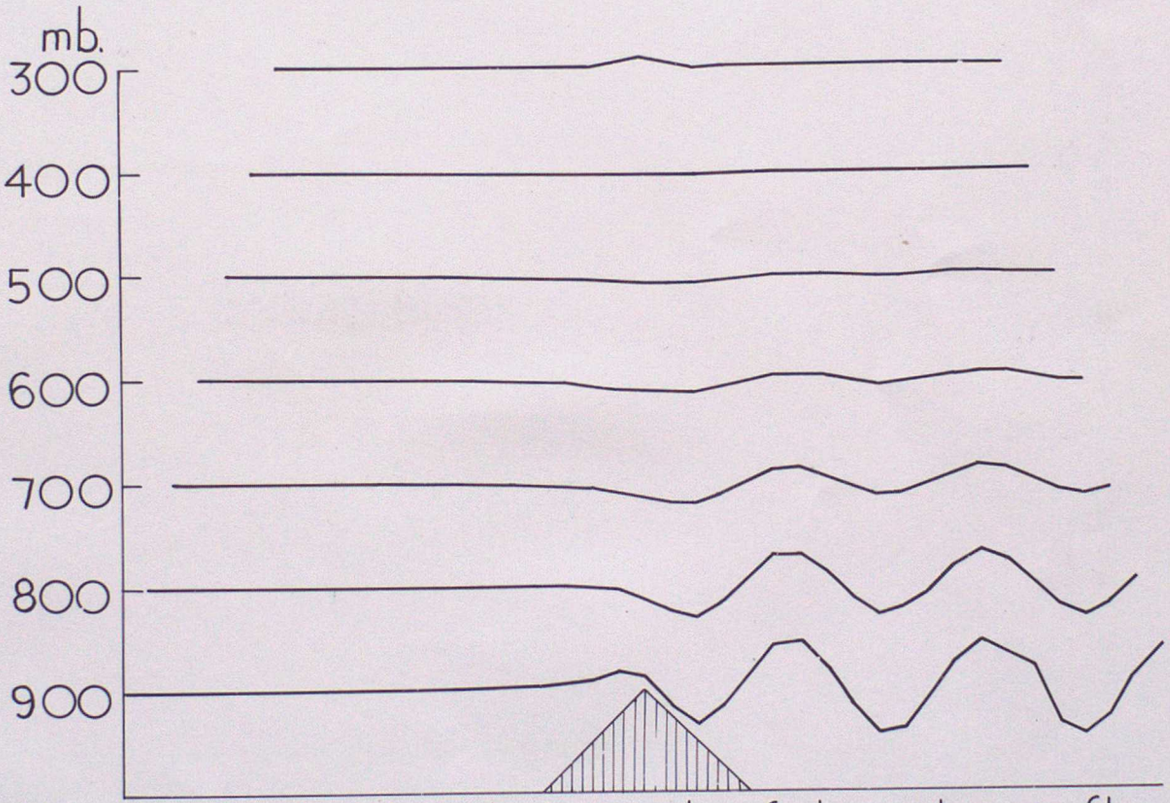




Example of calculated displacements of stream lines in flow over a ridge

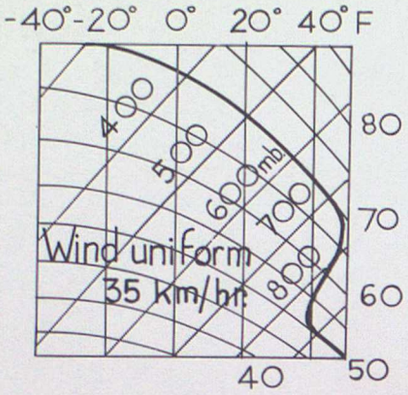
Fig. 7

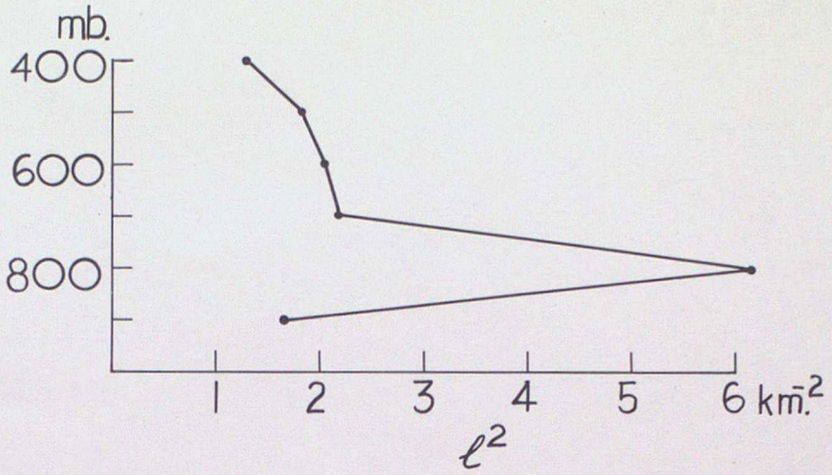


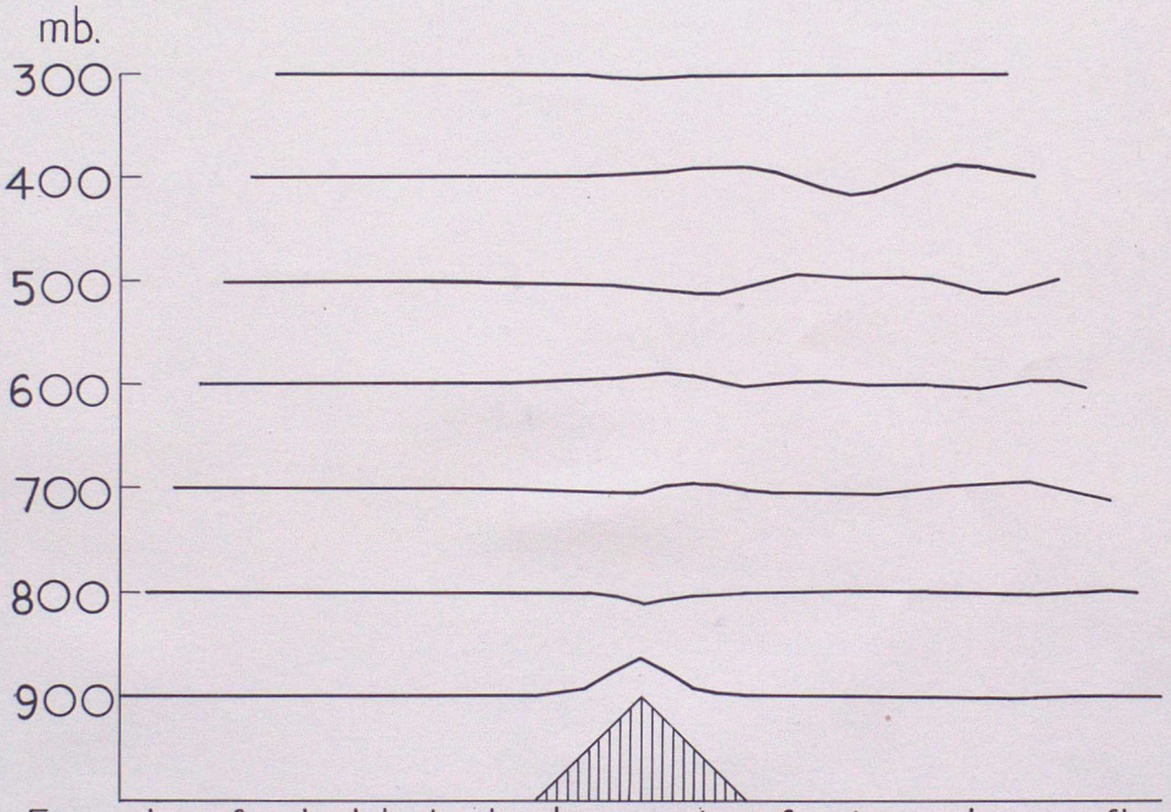


Example of calculated displacements of stream lines in flow over a ridge

Fig. 8

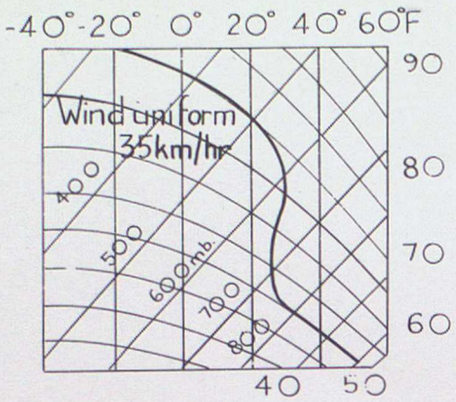


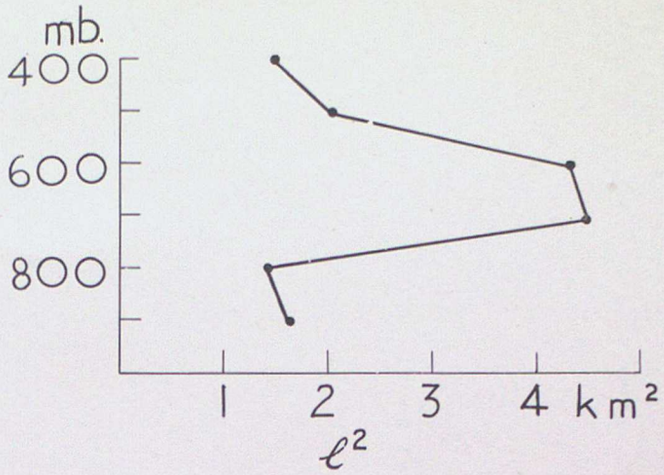


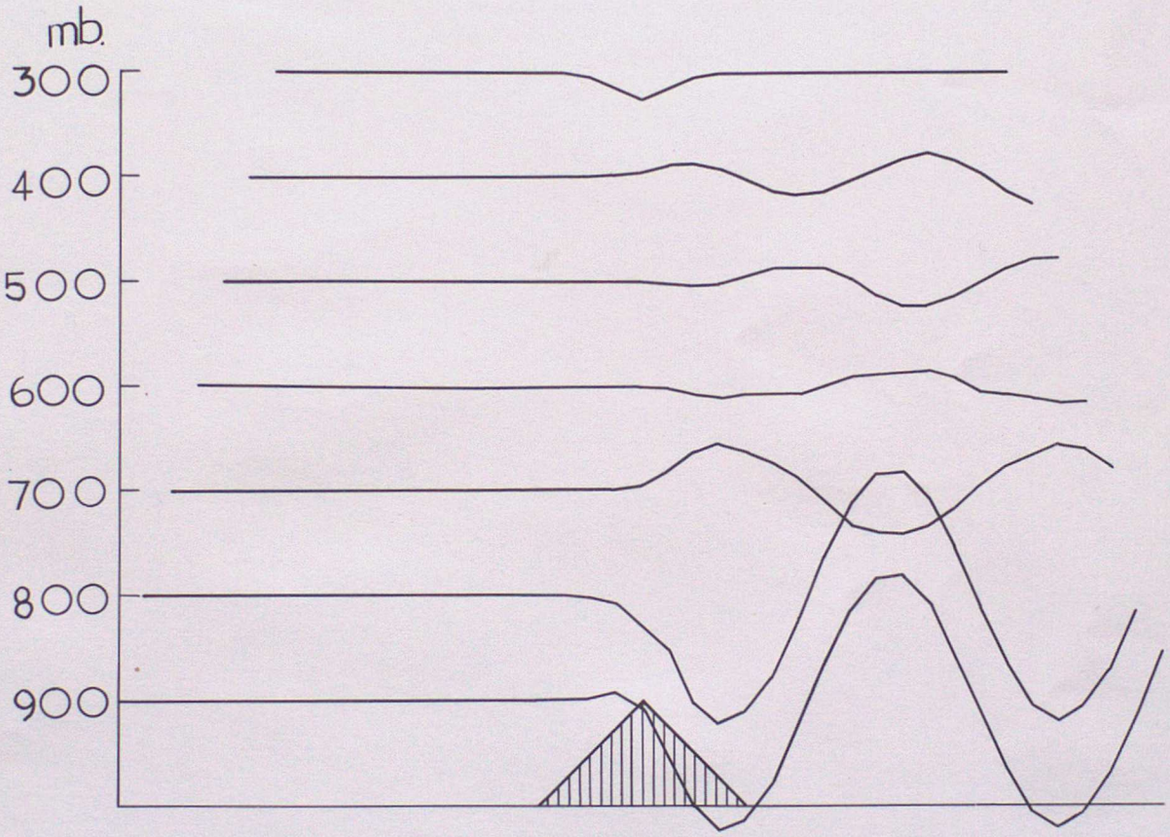


Example of calculated displacements of stream lines in flow over a ridge

Fig. 9







Example of calculated displacements of stream lines in flow over a ridge

Fig. 10

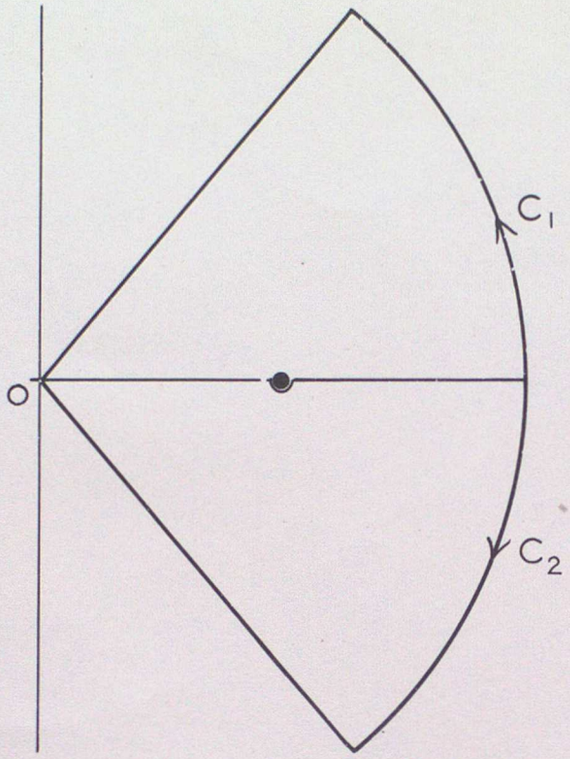


Fig. 11(a)

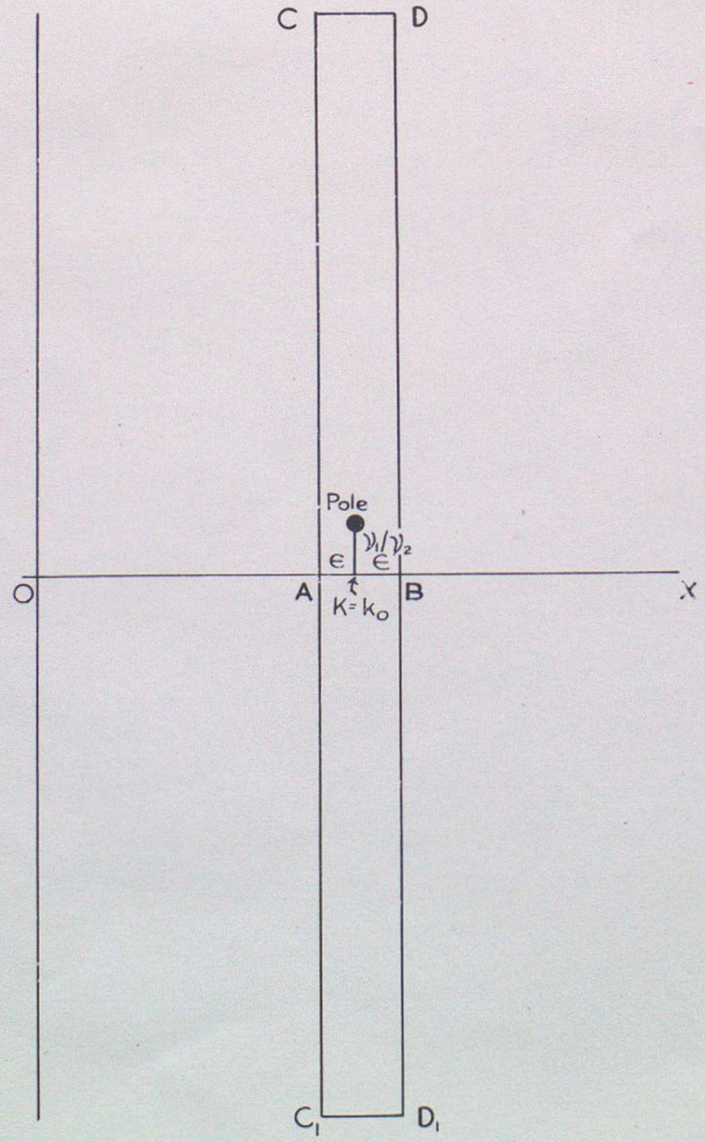


Fig. 11(b)